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Analysis for the optimal performance of three-channel split-flow heat exchangers

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Abstract—Governing equations for describing the axial variation of temperature differences of fluids flowing in a three-channel single-pass heat exchanger are formulated by adopting similar assumptions as those used in the classical log-mean-temperature-difference (LMTD) method for two-channel heat exchangers. A special-case solution and a generalized solution of these governing differential equations are obtained for designing exchangers with split-flow channels in both parallel-flow and counterflow configurations. The special-case solution can be obtained under the condition of having identical axial-temperature distributions in the split-shell-flow channels and is similar to the classical formulation for two-channel heat exchangers, but with some parameter modifications. Solutions of this general model confirm that the special-case model represents the optimum design of such heat exchangers. These results are also verified experimentally using a concentric-tube heat exchanger. Theoretically predicted heat-exchanger effectivenesses are found on the average to be within ±5% of the experimental measurements.

1. INTRODUCTION

The primary function of conventional heat exchangers is to transfer thermal energy from a hot fluid to a cold fluid. Although it is possible to have more than two flowing fluids present in a heat exchanger, three-channel heat exchangers have not been popular because classical design methods such as the log-mean-temperature-difference (LMTD) method or the number-of-transfer-unit (NTU) method are only valid for two-channel heat exchangers. The primary advantage of having a third flow channel is to increase the heat transfer capability by transferring thermal energy in two opposite directions, either from a hot fluid flowing in the two side channels to a cold fluid flowing in the central channel, or vice versa. Also, the effectiveness of a three-channel heat exchanger can be significantly greater than that of a two-channel heat exchanger if one of the two flows in the two-channel heat exchanger is split into two side flows on each side of the central channel because it increases the convective heat transfer surface areas between the flowing fluids.

Duvan [1] was granted a U.S. patent for inventing a three-channel single-pass split-flow heat exchanger. However, based on the authors' knowledge, methods for analyzing the heat transfer behaviour of heat exchangers of this type have never been reported in the open literature. Iqbal and Stachiewiez [2] presented a closed-form solution for heat transfer characteristics of mixed and split shell-flow heat exchangers in the cross-flow configuration based on the NTU method; however, their approach cannot be applied to three-channel exchangers because of the different nature of the problem they had solved. Three-channel heat exchangers basically consist of unmixed flows in three different channels where two different heat transfer paths in opposite directions are possible for each fluid. The governing equations for describing the heat-transfer behaviour of these heat exchangers are considerably different from those used in either the classical LMTD method or the NTU method. Prasad [3] analyzed the heat transfer problem of a two-channel double-pipe heat exchanger with a non-adiabatic condition at the outer surface. Although his governing equations are similar in form to those for a three-channel heat exchanger, they cannot be applied directly to three-channel heat exchangers because only two-flowing fluids were considered in his analysis, and the ambient fluid temperature at the outer surface was assumed to be constant. In view of the need to fully understand the heat transfer behaviour of three-channel heat exchangers, the primary objective of this work is to develop an analytical technique for designing this type of exchangers such that an optimal heat transfer capability can be achieved.

To derive the governing equations for the problem, similar assumptions as those used in the classical LMTD or NTU method are adopted in the present analysis. As was characterized by Taborek [4], the classical LMTD and NTU methods only deal with the thermodynamic part of the problem because they assume the overall heat-transfer coefficient is both known and constant. The same drawback also applies to the present analysis because similar assumptions are used to solve the problem for three-channel heat exchangers. Although the classical LMTD and NTU methods are not perfect, no other method has been formulated to replace their roles in heat exchanger design and analysis to this date. Therefore, the present method can also have the same importance to the

NOMENCLATURE

- A undetermined coefficient in (39) or total heat exchange area in (A2)
- A_{1i} heat transfer area between channels no. 1 and no. i ($i = 2,3$)
- a non-dimensional parameters defined in (10), (12), (13), (34) and (63)
- B undetermined coefficient in (39)
- b non-dimensional parameters defined in (34), (35), (56) and (57)
- C flow capacity rates defined in (7), (17) and (A1)
- c_{pi} specific heat of the fluid in channel no. i
- D_i coefficients in (B4) and (B9) ($i = 1,2,3$)
- L total length of the exchanger
- m_i mass flow rate of the flow in channel no. i
- Ntu equivalent numbers of transfer units defined in (20) and (67)
- P_{1i} heat transfer perimeter between channels no. 1 and no. i ($i = 2,3$)
- $\dot{Q}$ total heat transfer rate of the exchanger
- Q_{1i} heat transfer rate between flows in channels no. 1 and no. i ($i = 2,3$)
- s equivalent split-flow temperature
- s_i parameters defined in (11) ($i = 1,2$)
- T_i fluid temperature in channel no. i ($i = 1,2,3$)
- T_s equivalent split-shell-flow temperature
- U overall heat transfer coefficient
- U_{1i} overall heat transfer coefficient between channels no.1 and no. i ($i = 2,3$)
- u temperature difference defined in (10)
- v temperature difference defined in (10)
- x axial coordinate of a point in the exchanger
- z normalized axial coordinate defined in (10).

Greek symbols

- α non-dimensional parameter defined in (39)
- β non-dimensional parameter defined in (39)
- γ non-dimensional parameter defined in (40)
- ε heat exchanger effectiveness defined in (19)
- λ flow capacity ratio defined in (16)
- μ flow capacity ratio defined in (21)
- ν parameter defined in (B8)
- θ non-dimensional parameter defined in (63)
- ϕ non-dimensional parameter defined in (63)
- σ temperature difference defined in (C8)
- τ temperature difference defined in (C5)
- Δ temperature difference defined in (C5)
- ω temperature difference defined in (C14)
- η temperature difference defined in (C14).

Subscript

- A temperature difference at $x = 0$
- c properties of the cold fluid
- e equivalent values
- h properties of the hot fluid
- i temperature at the inlet
- L temperature difference at $x = L$
- o temperature at the outlet
- s equivalent properties of the split flow.

optimum design of three-channel heat exchangers as that of the classical methods to the design of the traditional two-channel heat exchangers.

Two mathematical approaches similar to those for the classical LMTD and the NTU methods are undertaken in the present analysis: one is to formulate the problem into ordinary differential equations such that outlet temperatures can be determined directly from their exact solutions; the other is to utilize these solutions to develop relationships between the heat exchanger effectiveness and an equivalent number of transfer units. The former case can be applied to any general three-channel heat exchangers; however, the latter technique is feasible only for split-shell-flow exchangers. Although the exact solutions of the governing differential equations for a generalized heat exchanger can be obtained, a special-case condition of having identical axial-temperature distributions in the split-shell-flow channels will also be investigated. Under such a condition, the governing equations become similar in form to those for two-channel heat exchangers and their solution can be formulated into the classical solution with some parameter modifications. Consequently, heat transfer characteristics calculated by using the special-case solution can be compared to those determined by solving the governing differential equations for the general case to examine the performance of this type of heat exchanger. As it turns out, this special case solution represents the optimum flow configuration for achieving maximum heat exchanger effectiveness.

To verify the predictive capability of the general-case theory, experimental measurements for various flow configurations for a three-channel split-flow

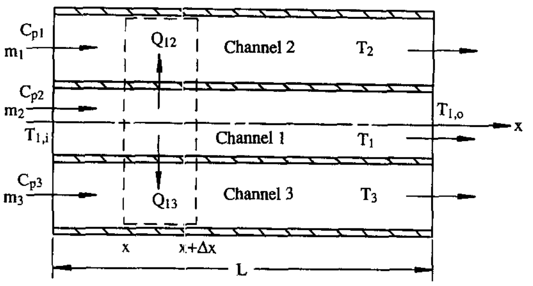

Fig. 1. Schematic of a three-channel single-pass split-flow heat exchanger.

single-pass concentric-tube heat exchanger were also obtained. Fluid flows in these exchangers were maintained steady and laminar. The choice of having steady laminar flows in concentric-tube heat exchangers is primarily based on the availability of accurate empirical correlations for calculating convective heat transfer coefficients for this type of flows. Similar to the classical LMTD or NTU method, the accuracy of the present techniques depends greatly upon the accuracy of estimating convective heat transfer coefficients. Kays [5] presented correlations of Nusselt numbers for laminar flows in a concentric-tube annulus. Although the correlations presented are only for specific values of radius ratios, these data are far more complete than what appears to exist in any other available literature. Therefore, Kays' [5] correlations are used to determine convective heat-transfer coefficients for the present theoretical computations. Values of the convective heat transfer coefficients for annular flows with radius ratios different from those specific radius ratios given by Kays [5] are obtained by using the method of graphical interpolation.

2. FORMULATION OF THE THEORETICAL MODEL

Although the analysis could be applied to any general three-channel-flow configuration, for the purposes of this paper because of its practicality, the problem under consideration is the determination of the heat transfer characteristics of three-channel single-pass split-flow heat exchangers as shown schematically in Fig. 1. The shell fluid is split to flow in the two side channels (channels no. 2 and no. 3) and the unsplit tube fluid flows in the central channel (channel no. 1), which is also a reference channel. If the split flows in the two side channels are in the same direction as the unsplit central flow in the reference channel, as shown in Fig. 1, the configuration will be considered parallel-flow. If the split flows are in the opposite direction of the central-channel reference flow, the exchanger will be considered counterflow. The geometry of channel cross-sections can either be annular, circular, rectangular, or of any arbitrary shape, as long as a common boundary exists between two adjacent channels.

2.1. *Differential equations governing the axial temperature distribution*

The mass flow rates and specific heats of fluids in channels numbered 1, 2 and 3 as shown in Fig. 1 are denoted as m_1, m_2, m_3, and c_{p1}, c_{p2}, c_{p3}, respectively. Temperatures of the flowing fluids in these channels are also designated as T_1, T_2 and T_3, respectively. Accordingly, the inlet and the outlet temperatures of these fluid flows are denoted as T_{1i}, T_{2i}, T_{3i} and T_{1o}, T_{2o}, T_{3o}. The total length of the exchanger is designated as L and the axial coordinate of any point in the heat exchanger is denoted as x, with $x = 0$ being located at the inlet of the central channel, channel no. 1. The overall heat transfer coefficient for the transverse heat transfer rate between fluids in channel no. 1 and channel no. 2, Q_{12}, is specified as U_{12} and that for the transverse heat transfer between fluids in channel no. 1 and channel no. 3, Q_{13}, is denoted as U_{13}. These coefficients are based upon reference surface areas, A_{12} and A_{13}, which can be represented by the multiplication of their corresponding reference perimeters, P_{12} and P_{13}, with the heat exchanger length, L. As a reference for the flow directions, the flow in channel 1 will always be in the positive x-direction. Assumptions used in the classical LMTD method [6] are adopted in the present analysis to derive the governing equation for describing the heat transfer behaviour of these heat exchangers.

The differential equation governing the axial temperature distribution for the fluid in a specific channel can be formulated by applying the steady-state form of the conservation of energy principle to an arbitrary control volume with length Δx, located between x and $x+\Delta x$. Axial conduction in the fluids and in the channel walls is considered negligible. Neglecting any change in kinetic as well as potential energies, and assuming that the heat transfer only takes place between the two fluids in the three channels, one can obtain an expression indicating the rate of enthalpy change of the fluid flowing through the control volume being equal to its rate of heat transfer. Thus, for the fluid flow in channel no. 1, the conservation of energy principle yields

$$m_1 c_{p1}[T_1(x+\Delta x) - T_1(x)] = -\int_x^{x+\Delta x} U_{12}P_{12}(T_1 - T_2)\,dx - \int_x^{x+\Delta x} U_{13}P_{13}(T_1 - T_3)\,dx. \quad (1)$$

Flow no. 1 exchanges heat with the fluids in both channels no. 2 and no. 3. Equation (1) is a general expression and can be applied to any cases with heat transfer being either from fluid no. 1 to fluids no. 2 and no. 3, or vice versa, because of its sign consistency.

Similar expressions for the split flows in channels no. 2 and no. 3 can be obtained as follows by recognizing that each of them exchanges heat only with fluid no. 1:

$$m_2 c_{p2}[T_2(x+\Delta x) - T_2(x)] = \pm \int_x^{x+\Delta x} U_{12}P_{12}(T_1 - T_2)\,dx \quad (2)$$

$$m_3 c_{p3}[T_3(x+\Delta x) - T_3(x)] = \pm \int_x^{x+\Delta x} U_{13}P_{13}(T_1 - T_3)\,dx. \quad (3)$$

As was mentioned earlier, if the flow direction of the split flows in channels no. 2 and no. 3 is the same as that of flow no. 1 (the positive x-direction), the exchanger has a parallel-flow configuration, and the signs on the right-hand side of equations (2) and (3) are positive. If the flow direction of fluids no. 2 and no. 3 is opposite to that of fluid no. 1 (the negative x-direction), the arrangement is referred to as a counterflow configuration, and the negative signs should be chosen for the right-hand side of equations (2) and (3).

Each of the above equations can be converted into a differential equation by dividing them by the length of the control volume, Δx, and then taking the limit as Δx approaches zero. The resulting coupled differential equations for each of the respective flow channels are expressed as follows:

(i) Channel 1

$$\frac{dT_1}{dx} = -\frac{U_{12}P_{12}}{C_1}(T_1 - T_2) - \frac{U_{13}P_{13}}{C_1}(T_1 - T_3) \quad (4)$$

(ii) Channel 2

$$\frac{dT_2}{dx} = \pm\frac{U_{12}P_{12}}{C_2}(T_1 - T_2) \quad (5)$$

(iii) Channel 3

$$\frac{dT_3}{dx} = \pm\frac{U_{13}P_{13}}{C_3}(T_1 - T_3) \quad (6)$$

where the flow capacity rate, $C = m\,c_p$, for each of the flow channels is given respectively by

$$C_1 \equiv m_1 c_{p1} \quad C_2 \equiv m_2 c_{p2} \quad C_3 \equiv m_3 c_{p3}. \quad (7)$$

Combining equations (4)–(6), one can obtain

$$\frac{du}{dz} + a_1 u + a_2 v = 0 \quad (8)$$

$$\frac{dv}{dz} + a_4 v + a_3 u = 0. \quad (9)$$

Parameters in these equations are defined as follows:

$$u \equiv T_1 - T_2 \quad v \equiv T_1 - T_3$$
$$z \equiv x/L \quad a_2 \equiv (U_{13}P_{13}L)/C_1$$
$$\text{and} \quad a_3 \equiv (U_{12}P_{12}L)/C_1. \quad (10)$$

Ratios of the flow capacity rates can also be defined as

$$s_1 \equiv C_1/C_2 \quad \text{and} \quad s_2 \equiv C_1/C_3. \quad (11)$$

Thus, the coefficients a_1 and a_4 can be expressed in terms of a_2 and a_3 for each different flow configuration as

(i) Parallel flows

$$a_1 \equiv a_3(1+s_1) \quad \text{and} \quad a_4 \equiv a_2(1+s_2) \quad (12)$$

(ii) Counterflows

$$a_1 \equiv a_3(1-s_1) \quad \text{and} \quad a_4 \equiv a_2(1-s_2). \quad (13)$$

For a three-channel single-pass split-flow heat exchanger, it is possible to express the total heat transfer rate and the effectiveness of the exchanger in a manner similar to those of a standard two-channel heat exchanger. Treating the flows in channel 2 and channel 3 as the split shell flows, one can express the total heat exchanger heat transfer rate, $\dot{Q}$, in terms of an equivalent shell-flow temperature, T_s, as

$$\dot{Q} = U_e P_e L \int_0^1 (T_1 - T_s)\,dz. \quad (14)$$

The equivalent heat transfer coefficient, U_e, is based on an equivalent perimeter, P_e. The equivalent shell-flow temperature can be related to temperatures of the fluids in the two split-flow side channels as

$$T_s = \lambda T_2 + (1-\lambda)T_3 \quad (15)$$

where the parameter, λ, is defined as

$$\lambda \equiv \frac{C_2}{C_s}. \quad (16)$$

The total flow capacity rate of the split shell flow is defined as

$$C_s \equiv C_2 + C_3. \quad (17)$$

Utilizing the right-hand side of equation (1), and the parameters defined in equation (10), one can express the total heat transfer rate as

$$\dot{Q} = U_{12}P_{12}L \int_0^1 u\,dz + U_{13}P_{13}L \int_0^1 v\,dz. \quad (18)$$

The effectiveness of the heat exchanger, ε, can be defined in the same manner as that of a two-channel heat exchanger as

$$\varepsilon \equiv \frac{\dot{Q}}{C_{min}(T_{1i} - T_{si})}. \quad (19)$$

The minimum flow capacity ratio, C_{min}, is either C_1 or C_s, whichever has the smaller value. Since fluids no. 2 and no. 3 are the two branches of the split shell flow, their inlet temperatures are assumed to be the same, and therefore, are equal to the inlet temperature of the equivalent shell flow, T_{si}. One can also define an equivalent number of heat transfer units in the same way as that of a two-channel heat exchanger to be

$$Ntu \equiv \frac{U_e P_e L}{C_{min}}. \quad (20)$$

To derive the relationship between the effectiveness and the number of transfer units, one can define the flow capacity ratio as $\mu \equiv C_{min}/C_{max}$. Hence, this parameter can be evaluated as

$$\mu = \frac{C_1}{C_s} \quad \text{if} \quad C_1 \leqslant C_s \quad \text{and} \quad \mu = \frac{C_s}{C_1} \quad \text{if} \quad C_1 \geqslant C_s. \tag{21}$$

2.2. *Special-case model*

Consider the special-case situation where

$$\frac{U_{12}P_{12}}{C_2} = \frac{U_{13}P_{13}}{C_3}. \tag{22}$$

If fluid no. 2 and fluid no. 3 have the same flow direction and the same inlet temperature, then it is clear from equations (5) and (6) that $T_2 = T_3$ for all values of x. Therefore, $u = v$, and consequently, equations (8) and (9) become

$$\frac{du}{dz} + (a_1 + a_2)u = \frac{dv}{dz} + (a_3 + a_4)v = 0. \tag{23}$$

Both equations (22) and (23) demand that

$$(a_1 + a_2) = (a_3 + a_4). \tag{24}$$

It is noted that the form of equation (23) is identical to that for the classical two-channel heat exchanger. Therefore, the classical LMTD and NTU approaches can be applied to the problem with slight modifications.

The solution of equation (23) for the case $a_1 + a_2 \neq 0$ can be determined as

$$u = v = u_A\, e^{-(a_1+a_2)z} \tag{25}$$

where $u_A = v_A$ are the temperature differences, u and v, evaluated at $x = 0$. Temperature differences at $x = L$ can also be expressed as

$$u_L = v_L = u_A\, e^{-(a_1+a_2)}. \tag{26}$$

The outlet temperatures of the flowing fluids can be determined by evaluating u_A for each configuration. For a split-flow heat exchanger, its heat transfer effectiveness can also be formulated in terms of the equivalent number of transfer units as well as the heat capacity ratio between the flow in the central channel and the combined split flow in the other two channels. These effectiveness–*Ntu* relationships for parallel-flow and counterflow configurations can be derived by utilizing the above general solution, the special-condition expressions of equations (18)–(22), and the following equation obtained for the special-case condition:

$$U_e P_e = U_{12}P_{12} + U_{13}P_{13}. \tag{27}$$

Summarized below are the expressions derived for determining the outlet temperatures and the heat-exchanger effectiveness for these two possible configurations.

(1) *Parallel-flow configuration.* Outlet temperatures of the flowing fluids can be determined as

$$T_{1o} = T_{1i} - \frac{(a_2+a_3)(T_{1i}-T_{2i})}{(a_2+a_3+a_3 s_1)}[1 - e^{-(a_2+a_3+a_3 s_1)}] \tag{28}$$

$$T_{2o} = T_{3o} = T_{1o} - (T_{1i} - T_{2i})\, e^{-(a_2+a_3+a_3 s_1)}. \tag{29}$$

The equivalent number of transfer units and the heat-exchanger effectiveness can also be derived to be

$$Ntu = \frac{(a_1+a_2)}{(1+\mu)} \tag{30}$$

$$\varepsilon = \frac{1}{(1+\mu)}[1 - e^{-(1+\mu)Ntu}]. \tag{31}$$

(2) *Counterflow configuration.* The temperature differences at $x = 0$ and $x = L$ can be expressed as

$$u_A = v_A = T_{1i} - T_{2o} = T_{1i} - T_{3o} \tag{32}$$

$$u_L = v_L = T_{1o} - T_{2i} = T_{1o} - T_{3i}. \tag{33}$$

Outlet temperatures can then be determined to be

$$T_{1o} = \frac{b_9 b_{10} - b_8 b_{11}}{b_{12}} \tag{34}$$

$$T_{2o} = T_{3o} = \frac{b_{11} - b_9}{b_{12}} \tag{35}$$

where

$$b_8 \equiv \frac{a_5}{a_6}(b_{10} - 1) \quad b_9 \equiv \frac{T_{1i}}{a_6}(a_5 b_{10} - a_3 s_1)$$

$$b_{10} \equiv e^{-a_6} \quad b_{11} \equiv T_{2i} + b_{10} T_{1i}$$

$$b_{12} \equiv b_{10} - b_8 \quad a_5 \equiv a_2 + a_3 \quad a_6 \equiv a_5 - a_3 s_1.$$

The equivalent number of transfer units can also be determined as

$$Ntu = \frac{(a_1+a_2)}{(1-\mu)} \quad \text{for } \mu \equiv \frac{C_1}{C_s} < 1 \tag{36a}$$

$$Ntu = \frac{(a_1+a_2)}{(\mu-1)} \quad \text{for } \mu \equiv \frac{C_s}{C_1} < 1 \tag{36b}$$

$$Ntu = \frac{a_2}{(1-\lambda)} \quad \text{for } \mu \equiv \frac{C_1}{C_s} = 1. \tag{36c}$$

The effectiveness of the heat exchanger can be derived to be

$$\varepsilon = \frac{1 - e^{(\mu-1)Ntu}}{1 - \mu\, e^{(\mu-1)Ntu}} \quad \text{for } \mu < 1 \tag{37a}$$

$$\varepsilon = \frac{Ntu}{1 + Ntu} \quad \text{for } \mu = 1. \tag{37b}$$

Equation (37b) is obtained from equation (37a) by using L'Hospital's rule.

It is significant that this special-case model, where the split-flow distribution satisfies the condition shown in equation (22), can be shown to represent the case of having the maximum effectiveness for a three-channel, split-flow heat exchanger. However, verification of this phenomenon requires comparing the numerical results obtained for this case to those obtained for the general case.

2.3. *Generalized model*

Equations (4) and (5) can be combined into the following second-order differential equation by eliminating the dependent variable v:

$$\frac{d^2u}{dz^2}+(a_1+a_4)\frac{du}{dz}+(a_1a_4-a_2a_3)u=0. \quad (38)$$

The general solution of equation (38) for the case $a_1a_4 \neq a_2a_3$ can be expressed as

$$u = e^{-\alpha z}(A\sinh\beta z+B\cosh\beta z) \quad (39a)$$

where $\alpha \equiv (a_1+a_4)/2$ and $\beta \equiv \sqrt{([(a_1-a_4)^2/4]+a_2a_3)}$. For a three-channel heat exchanger, both s_1 and s_2 cannot be zero. Hence, the situation of $a_1a_4 = a_2a_3$ can only exist for exchangers in the counterflow configuration. Under this special situation, the general solution becomes

$$u = A+B\,e^{-2\alpha z}. \quad (39b)$$

Undetermined constants, A and B, can be evaluated by using end conditions at inlets and outlets. The general solution of the dependent variable v can also be obtained as follows by substituting equation (39a) or equation (39b) into equation (8):

$$v=\frac{e^{-\alpha z}}{a_2}[(A\gamma-B\beta)\sinh\beta z+(B\gamma-A\beta)\cosh\beta z] \quad \text{for } a_1a_4 \neq a_2a_3 \quad (40a)$$

$$v=\frac{1}{a_2}(Ba_4\,e^{-2\alpha z}-Aa_1) \quad \text{for } a_1a_4 = a_2a_3 \quad (40b)$$

where $\gamma \equiv (a_4-a_1)/2$.

Boundary conditions in general can be expressed as

$$\text{At } z=0: u=u_A \quad \text{and} \quad v=v_A \quad (41)$$

$$\text{At } z=1: u=u_L \quad \text{and} \quad v=v_L. \quad (42)$$

Parameters u_A, v_A, u_L, and v_L depend upon the flow configuration of the heat exchanger. In general, these parameters are functions of inlet and outlet temperatures and consequently, they can also be undetermined variables. In any case, temperature differences u_A and v_A can either be evaluated from inlet temperatures directly or be determined by-manipulating equations (39) and (40) to satisfy inlet boundary conditions. Therefore, the general solution of temperature differences for the case $a_1a_4 \neq a_2a_3$ can be expressed in terms of u_A and v_A as

$$u=e^{-\alpha z}\left[u_A\cosh\beta z+\frac{1}{\beta}(\gamma u_A-a_2v_A)\sinh\beta z\right] \quad (43)$$

$$v=e^{-\alpha z}\left[v_A\cosh\beta z-\frac{1}{\beta}(a_3u_A+\gamma v_A)\sinh\beta z\right]. \quad (44)$$

For the case $a_1a_4 = a_2a_3$, the general solution becomes

$$u=\frac{1}{2\alpha}[(a_4u_A-a_2v_A)+(a_1u_A+a_2v_A)\,e^{-2\alpha z}] \quad (45)$$

$$v=\frac{1}{2\alpha}[(a_1v_A-a_3u_A)+(a_3u_A+a_4v_A)\,e^{-2\alpha z}]. \quad (46)$$

Temperature differences at $x = L$ for the case $a_1a_4 \neq a_2a_3$ can then be determined as

$$u_L=e^{-\alpha}\left[u_A\cosh\beta+\frac{1}{\beta}(\gamma u_A-a_2v_A)\sinh\beta\right] \quad (47)$$

$$v_L=e^{-\alpha}\left[v_A\cosh\beta-\frac{1}{\beta}(a_3u_A+\gamma v_A)\sinh\beta\right]. \quad (48)$$

Those for the case $a_1a_4 = a_2a_3$ can also be expressed as

$$u_L=\frac{1}{2\alpha}[(a_4u_A-a_2v_A)+(a_1u_A+a_2v_A)\,e^{-2\alpha}] \quad (49)$$

$$v_L=\frac{1}{2\alpha}[(a_1v_A-a_3u_A)+(a_3u_A+a_4v_A)\,e^{-2\alpha}]. \quad (50)$$

Outlet temperatures of fluid flows in all three channels can be determined by evaluating temperature differences at $x = 0$ using inlet temperatures of fluid flows in all three channels, T_{i1}, T_{i2}, and T_{i3}. Results for the parallel-flow and the counterflow configurations are summarized below.

(1) Parallel flow configuration ($a_1a_4 \neq a_2a_3$ always):

Temperature differences at $x = 0$ can be determined as

$$u_A = T_{1i}-T_{2i} \quad (51)$$

$$v_A = T_{1i}-T_{3i}. \quad (52)$$

Outlet temperatures of the flowing fluids can then be shown to be

$$T_{1o}=\frac{C_1T_{1i}+C_2(u_L+T_{2i})+C_3(v_L+T_{3i})}{C_1+C_2+C_3} \quad (53)$$

$$T_{2o} = T_{1o}-u_L \quad (54)$$

$$T_{3o} = T_{1o}-v_L. \quad (55)$$

Temperature differences at $x = L$ can be evaluated by using equations (47) and (48).

(2) Counterflow configuration:

Temperature differences at $x = 0$ can be determined by utilizing equations (47)–(50) to be

$$u_A=\frac{1}{b_1}(b_4b_6-b_3b_7) \quad (56)$$

$$v_A=\frac{1}{b_2}(b_2b_7-b_4b_5) \quad (57)$$

where for the case $a_1a_4 \neq a_2a_3$,

$$b_1 \equiv b_2b_6-b_3b_5$$

$$b_2 \equiv \cosh\beta+\frac{1}{\beta}(a_3+\gamma)\sinh\beta$$

$$b_3 \equiv \frac{1}{\beta}(\gamma - a_2)\sinh\beta - \cosh\beta$$

$$b_4 \equiv e^{\alpha}(T_{3i} - T_{2i})$$

$$b_5 \equiv \frac{1}{s_1} - e^{-\alpha}\left(\cosh\beta + \frac{\gamma}{\beta}\sinh\beta\right)$$

$$b_6 \equiv \frac{a_2}{\beta}e^{-\alpha}\sinh\beta + \frac{1}{s_2}$$

$$b_7 \equiv \left(\frac{1}{s_1} - 1\right)(T_{1i} - T_{2i}) + \frac{1}{s_2}(T_{1i} - T_{3i}).$$

For the case $a_1a_4 = a_2a_3$, parameters b_2–b_7 should be evaluated as

$$b_2 \equiv a_3 + a_4 - s_1 a_3 e^{-2\alpha}$$

$$b_3 \equiv s_2 a_2 e^{-2\alpha} - a_1 - a_2$$

$$b_4 \equiv 2\alpha(T_{3i} - T_{2i})$$

$$b_5 \equiv a_4 - \frac{2\alpha}{s_1} + a_1 e^{-2\alpha}$$

$$b_6 \equiv -a_2 - \frac{2\alpha}{s_2} + a_2 e^{-2\alpha}$$

$$b_7 \equiv 2\alpha\left[(T_{1i} - T_{2i})\left(1 - \frac{1}{s_1}\right) - \frac{1}{s_2}(T_{1i} - T_{3i})\right].$$

Outlet temperatures of the flowing fluids in three different channels can then be determined as

$$T_{1o} = T_{1i} + \frac{1}{s_1}(T_{2i} + u_A - T_{1i}) + \frac{1}{s_2}(T_{3i} + v_A - T_{1i}) \tag{58}$$

$$T_{2o} \equiv T_{1i} - u_A \tag{59}$$

$$T_{3o} \equiv T_{1i} - v_A. \tag{60}$$

The total heat transfer rate of the heat exchanger, $\dot{Q}$, can be determined from outlet temperatures as

$$\dot{Q} = C_1(T_{1i} - T_{1o}) = C_2(T_{2o} - T_{2i}) + C_3(T_{3o} - T_{3i}). \tag{61}$$

For a split-flow exchanger in the counterflow arrangement, the solution cannot be obtained by using the above analysis if the heat capacity ratio is exactly equal to unity ($\mu = 1$ and $C_t = C_s$). The expressions of temperature differences shown in equations (45) and (46) become trivial because $u_A = v_A = 0$ under this circumstance. A similar situation also happens to the classical LMTD solution for a two-channel counterflow exchanger when its heat capacity ratio is exactly equal to unity. This can be seen by examining the closed-form LMTD temperature relationships shown in Appendix A for conventional two-channel heat exchangers. The log mean temperature difference becomes meaningless and undeterminable under this circumstance. Therefore, the total heat transfer rate and outlet temperatures for this particular case can only be obtained by reformulating the governing equation for this special condition as shown in Appendix A or applying L'Hospital's rule to the general expression of the heat exchanger effectiveness for counterflow exchangers [see equation (37b)]. The technique of applying L'Hospital's rule to the similar case for a split-flow exchanger is infeasible because the nature of the formulation is different from that of a conventional two-channel heat exchanger. Therefore, the solution to the problem can only be obtained by reformulating the governing equations to satisfy this special-case condition. This theoretical analysis for three-channel split-flow heat exchangers with the counterflow configuration under the special condition of having the value of the flow capacity ratio being precisely equal to unity is presented in Appendix B.

As is shown in Appendix C, the number of transfer units of a split-shell-flow heat exchanger under a generalized condition can be expressed in terms of the solution of the governing equation of a three-channel heat exchanger as

$$Ntu = \left(\frac{C_1}{C_{\min}}\right)\left[\frac{a_3\theta + a_2\phi}{\lambda\theta + (1-\lambda)\phi}\right] \tag{62}$$

where

$$\theta \equiv \int_0^1 u\,dz \quad \text{and} \quad \phi \equiv \int_0^1 v\,dz.$$

Utilizing equations (43) and (44) to evaluate the above integrations, one can express the undetermined parameters, θ and ϕ, for the case $a_1a_4 \neq a_2a_3$ as

$$\theta = u_A\left[a_7\left(1 - \frac{\gamma}{\beta}\right) - a_8\left(1 + \frac{\gamma}{\beta}\right)\right] + \frac{v_A a_2}{\beta}(a_7 + a_8) \tag{63}$$

$$\phi = v_A\left[a_7\left(1 + \frac{\gamma}{\beta}\right) + a_8\left(\frac{\gamma}{\beta} - 1\right)\right] + \frac{u_A a_3}{\beta}(a_7 + a_8) \tag{64}$$

where

$$a_7 \equiv \frac{1}{2(\alpha+\beta)}[1 - e^{-(\alpha+\beta)}]$$

and

$$a_8 \equiv \frac{1}{2(\beta-\alpha)}[1 - e^{(\beta-\alpha)}].$$

For the case $a_1a_4 = a_2a_3$, these parameters become

$$\theta = \frac{1}{2\alpha}\left[a_4u_A - a_2v_A + \frac{1}{2\alpha}(a_1u_A + a_2v_A)(1 - e^{-2\alpha})\right] \tag{65}$$

$$\phi = \frac{1}{2\alpha}\left[a_1v_A - a_3u_A + \frac{1}{2\alpha}(a_3u_A + a_4v_A)(1 - e^{-2\alpha})\right]. \tag{66}$$

Based on the definition of the equivalent number of transfer units shown in equation (20), it can also be expressed as

$$Ntu = \frac{Ntu_1\theta + Ntu_2\phi}{\lambda\theta + (1-\lambda)\phi} \quad (67)$$

where

$$Ntu_1 \equiv \frac{U_{12}P_{12}L}{C_{min}} \quad \text{and} \quad Ntu_2 \equiv \frac{U_{13}P_{13}L}{C_{min}}.$$

These numbers of transfer units corresponding to partial heat transfer rates can be determined as

$$\text{If } \mu \equiv \frac{C_l}{C_s} \leqslant 1: \; Ntu_1 = a_3 \; \text{ and } \; Ntu_2 = a_2 \quad (68)$$

$$\text{If } \mu \equiv \frac{C_s}{C_l} \leqslant 1: \; Ntu_1 = \frac{a_3}{\mu} \; \text{ and } \; Ntu_2 = \frac{a_2}{\mu}. \quad (69)$$

As is shown in Appendix C, the relationship between the effectiveness and the equivalent number of transfer units for a three-channel single-pass split-flow heat exchanger in either the parallel-flow or the counterflow configuration can be derived to be in the same form as that of a classical two-channel heat exchanger [6, 7]. Therefore, using equations (21) and (67) to compute μ and Ntu, respectively, one can determine the heat-exchanger effectiveness by using an appropriate equation as those shown in equations (31) and (37) for the special-case model. However, for a counterflow three-channel split-flow heat exchanger having a heat capacity ratio being equal to unity, the value of the number of transfer units determined by using equation (67) becomes singular. As is shown in Appendix B, the proper Ntu value under this circumstance should be obtained from equation (B23) instead.

3. THEORETICAL RESULTS AND EXPERIMENTAL VERIFICATION

To verify the present theory, heat transfer characteristics of a three-channel split-flow single-pass concentric-tube heat exchanger are experimentally measured and compared with those calculated theoretically. A schematic of the three-channel split-flow heat exchanger and the associated instrumentation used for the experimental measurements is shown in Fig. 2. It is constructed by assembling three concentric copper tubes and an inner solid copper wire. Figure 3 depicts a cross-section of the three-channel concentric-tube heat exchanger. The outer diameters of the inner, the central and the outer tubes are 6.35, 9.53 and 12.7 mm, respectively. The wall thickness of these three tubes are 0.762, 0.762 and 0.889 mm, respectively. The diameter of the inner solid wire is 1.59 mm and the length of the heat exchanger is 1.22 m. (The lengths of the end fittings are small compared to the overall length, L, of the heat exchanger.) Both parallel-flow and counterflow configurations were tested by using the same heat exchanger. The central annulus is designated as channel 1 in which the unsplit fluid flows. The split-flow fluid flows in both the outer and the inner annular channels, which are designated as channel 2 and channel 3, respectively. The split-flow fluid in these channels have the same inlet temperatures. Water was the fluid used for all three channels. Volumetric flowrates for each channel were measured by using three separate variable area flowmeters. Inlet and outlet fluid temperatures were measured using thermocouples and a multiple-channel strip-chart temperature recorder. Flowrates in all three channels were kept within the range of laminar flows to avoid transition. An effort was made to minimize entrance effects by having the water enter all three channels from a polyvinyl chloride tubing of about the same size as that of the heat exchanger channel.

Tables 1 and 2 summarize the comparison of experimental results to the theoretical predictions calculated by using the generalized model for parallel-flow and counterflow configurations, respectively. Two different values of the heat exchanger effectiveness are calculated from the experimental measurements: the effectiveness calculated from the heat transfer rate in the central unsplit annular-tube-flow channel, ε_l; and the effectiveness calculated from the heat transfer rates in the inner and the outer split-flow channels, ε_s. Theoretical results are obtained by utilizing the correlations of Nusselt numbers and influence coefficients for fully-developed laminar annular flows reported by Kays [5]. Since these correlations were only available for a limited number of values of radius ratios, graphical interpolation was used to estimate actual values of Nusselt numbers and influence coefficients. In general, results shown in Tables 1 and 2 indicate that theoretical predictions are on the average within $\pm 5\%$ of the experimental measurements. Results for the case of having no-flow in one of the split-flow channels are also included. Due to computational procedures, the present theory will become invalid if one of the three channels has a zero flowrate, therefore, theoretical results for these no-flow cases are obtained by using a relatively small flowrate in this no-flow channel. Discrepancies for these cases between the theoretical results and experimental measurements are approximately on the order of $\pm 18\%$. Values of Reynolds numbers and heat capacity ratios used for theoretical calculations of these cases are included in parentheses. Except for the case of $\lambda = 0$, the theoretical data shown agree within $\pm 5\%$ with the experimental measurements. Since Reynolds numbers of the flow in the inner annulus (channel no. 3) are relatively large for cases with $\lambda = 0$, the laminar-flow correlations of Nusselt numbers might not be accurate, or the fluid in the channel might be experiencing some degree of turbulence. Consequently, for $\lambda = 0$, the values of the effectiveness predicted by using the present theoretical model are lower than experimental measurements for both the parallel-flow and the counterflow configurations.

An important observation can be made about the heat exchanger effectiveness, ε, from both the experimental and the theoretical data, and for both con-

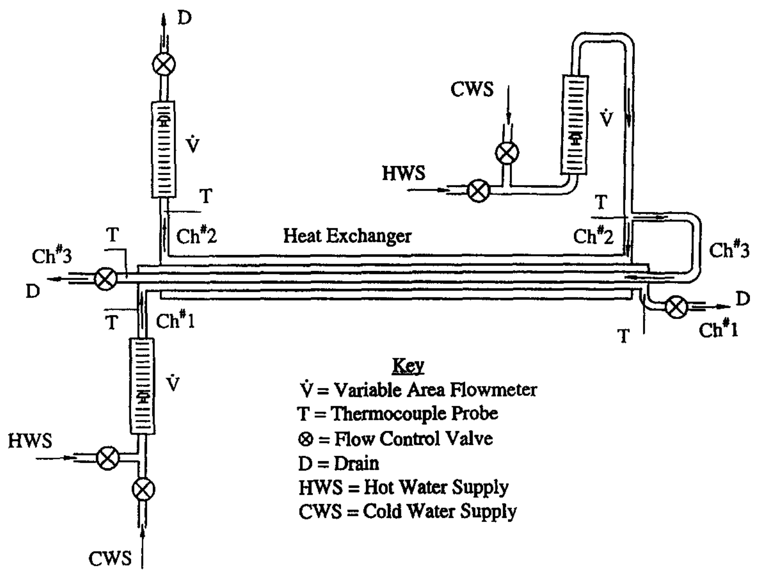

Fig. 2. Schematic of the experimental apparatus associated with testing the three-channel split-flow concentric-tube heat exchanger; counterflow configuration (for parallel flow, flowmeter at channel no. 1 left is moved to channel no. 1 right).

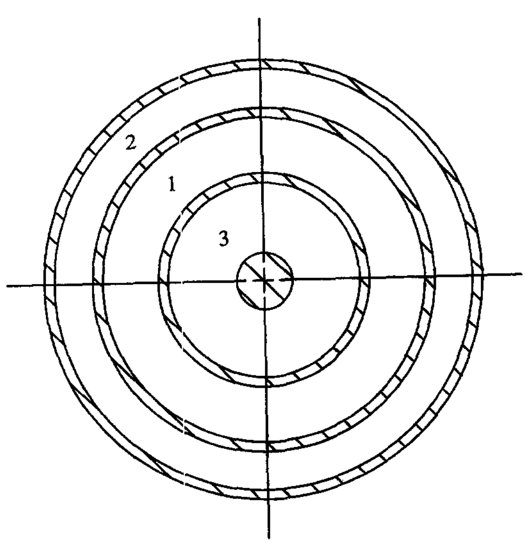

Fig. 3. Cross-section of a three-channel split-flow concentric-tube heat exchanger.

figurations, although it is more pronounced for the counterflow results. The observation is that heat exchanger effectiveness is lower for $\lambda = 0$ or $\lambda = 1$ (no-flow in channel 2 or channel 3, respectively), than it is for values of λ in between. This suggests that there is some optimal split-flow distribution parameter, λ, where the heat exchanger effectiveness is a maximum. This will be developed in the next section.

4. VERIFICATION THAT THE SPECIAL-CASE MODEL REPRESENTS THE OPTIMUM DESIGN

The focus of this paper is to establish the fact that the flow distribution of the split flow is an important parameter in determining the overall performance of a three-channel single-pass split-flow heat exchanger. In fact, as suggested by the data presented in the previous section, there is an optimal split-flow distribution, λ, which will maximize the heat-exchanger effectiveness. Furthermore, this optimal split-flow distribution will be shown to be the same as that specified by the special-case model presented earlier. This will be done by utilizing the generalized model to determine the heat exchanger effectiveness, ε, as a function of the split-flow distribution parameter, λ, for several different flow capacity ratios, C_1/C_s.

The geometry and dimensions of the heat exchanger analyzed are identical to those of the experimental apparatus discussed previously. The volumetric flow rate of the non-split central annular flow is assumed to be 0.613 l h^{-1}, while those of the split-shell flows are varied to yield the desired values of λ and C_1/C_s under the laminar-flow regime. Inlet temperatures of the two split flows and the central annular flow are 12.2, 12.2 and 47.8°C, respectively. Data corresponding to cases which have either turbulent or transitional flow in one of the three channels are excluded so a proper comparison can be made. Figures 4 and 5 correlate the predicted values of the heat exchanger effectiveness with the heat capacity ratios of the split-flow exchanger, λ and C_1/C_s, for parallel-flow and counterflow configurations, respectively. The results for the case of $C_1/C_s = 1$ shown in Fig. 5 are calculated by using equations derived in Appendix B for counterflow heat exchangers. All of the curves shown in Figs. 4 and 5 demonstrate the existence of a maximum value of the effectiveness at a given value of split-flow distribution parameter, λ, for each flow capacity ratio, C_1/C_s. Values of the heat-exchanger effectiveness predicted, using the special-case model

Table 1. Comparison of theoretical results with experimental measurements for a single-pass, three-channed split-flow concentric-tube heat exchanger in parallel-flow configuration

							Experimental measurements					Present analysis			
Re_1	Re_2	Re_3	C_1/C_s	λ	T_{1i} [°C]	T_{si} [°C]	T_{1o} [°C]	T_{2o} [°C]	T_{3o} [°C]	ε_1	ε_s	T_{1o} [°C]	T_{2o} [°C]	T_{3o} [°C]	ε
	0		0.993	0									—		
1364	(11)	1743†	(0.975)	(0.018)	47.2	11.1	31.9	—	25.6	0.423	0.403	35.2	(35.3)	22.6	0.337
1354	306	915	0.993	0.475	47.8	11.1	30.6	30.3	26.7	0.470	0.473	30.7	31.1	25.3	0.479
1345	406	621	0.993	0.650	47.5	11.1	30.0	28.1	28.1	0.481	0.469	29.8	29.3	27.7	0.488
1364	492	382	0.993	0.790	47.8	11.1	30.3	27.8	30.0	0.477	0.471	29.9	28.4	30.2	0.488
		0	0.993	1										—	
1371	619	(22)	(0.980)	(0.988)	48.1	11.1	31.7	27.2	—	0.444	0.439	30.7	28.1	(30.8)	0.471

† Note that there could have been some turbulence present in this tube. This would account for the performance being better than predicted.

Table 2. Comparison of theoretical results with experimental measurements for a single-pass, three-channel split-flow concentric-tube heat exchanger in a counterflow configuration

							Experimental measurements					Present analysis			
Re_1	Re_2	Re_3	C_1/C_s	λ	T_{1i} [°C]	T_{si} [°C]	T_{1o} [°C]	T_{2o} [°C]	T_{3o} [°C]	ε_1	ε_s	T_{1o} [°C]	T_{2o} [°C]	T_{3o} [°C]	ε
	0		1.006	0									—		
1344	(29)	1752†	(0.961)	(0.045)	47.5	11.1	30.0	—	28.1	0.478	0.466	33.4	(47.1)	23.6	0.387
1270	335	959	0.996	0.475	47.5	11.1	24.2	38.3	30.3	0.641	0.634	24.9	39.7	28.2	0.621
1344	464	592	0.994	0.690	48.1	10.8	24.2	34.4	34.4	0.642	0.638	26.3	35.3	34.0	0.650
1281	527	399	0.995	0.800	47.8	11.1	24.7	32.2	38.1	0.629	0.611	24.3	33.4	38.7	0.641
		0	1.006	1										—	
1331	645	(57)	(0.976)	(0.970)	48.1	12.2	28.3	30.3	—	0.547	0.504	26.8	32.5	(46.8)	0.593

† Note that there could have been some turbulence present in this tube. This would account for the performance being better than predicted.

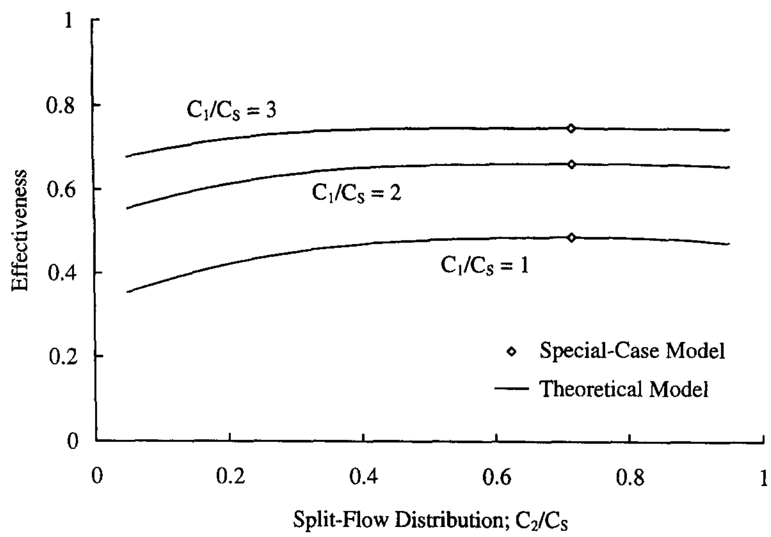

Fig. 4. Influence of split-flow distribution on performance of a three-channel parallel-flow heat exchanger.

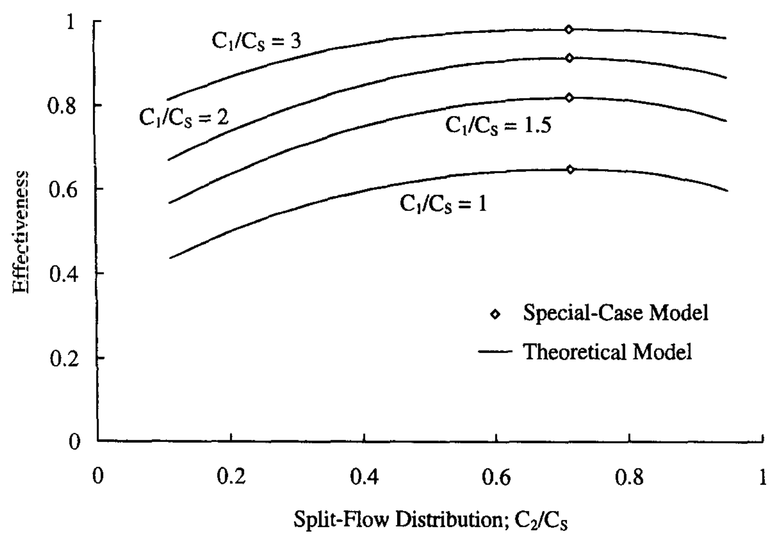

Fig. 5. Influence of split-flow distribution on performance of a three-channel counterflow heat exchanger.

shown in equations (31) and (37), which is valid for the special case conditions specified by equation (22), are also depicted in Figs. 4 and 5. It is clear from the data that the special-case model represents the optimal design for the three-channel heat exchanger effectiveness. Using equations (16), (17) and (22), one can derive the optimal value of the split-flow distribution parameter, λ_{op}, to be

$$\lambda_{op} = \frac{1}{\left(1+\dfrac{U_{13}P_{13}}{U_{12}P_{12}}\right)}. \tag{70}$$

It should be noted that for turbulent flows, the above equation would be transcendental, since U_{12} and U_{13} would be a function of the split-flow distribution parameter, λ, as well as the flow capacity ratio, μ.

5. CONCLUSIONS

Governing equations and their closed-form solutions for determining the heat transfer characteristics of three-channel single-pass split-flow heat exchangers have been successfully developed for both parallel-flow and counterflow configurations. A special-case model and the generalized model expressing the relationships between the heat exchanger effectiveness and the equivalent number of transfer units have been obtained. The special-case model is similar to that of the classical two-channel heat exchanger, but with some parameter modifications. Using the generalized model for a three-channel concentric-tube configuration, the special-case model has been shown to represent the optimum design of such heat exchangers for both parallel flows and counterflows. For the same concentric-tube configurations, values of the heat-exchanger effectiveness predicted by using the present theory are shown to agree within $\pm 5\%$ of the experimental measurements. Therefore, the theory has been successfully verified.

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APPENDIX A: CLOSED-FORM LMTD TEMPERATURE RELATIONSHIPS FOR CONVENTIONAL TWO-CHANNEL HEAT EXCHANGERS

The classical LMTD formulation can be used to obtain closed-form expressions of outlet temperatures of conventional two-channel heat exchangers. The heat balance between the hot fluid with temperature T_h and the cold fluid with temperature T_c can be expressed as

$$\dot{Q} = C_h(T_{hi}-T_{ho}) = C_c(T_{co}-T_{ci}). \quad \text{(A1)}$$

Here, C_h and C_c are the flow capacity rates of the hot and the cold fluids, respectively. For the case of $C_h \neq C_c$, the LMTD formulation of the total heat transfer rate for the counterflow configuration is

$$\dot{Q} = UA\left[\frac{(T_{ho}-T_{ci})-(T_{hi}-T_{co})}{\ln\left(\dfrac{T_{ho}-T_{ci}}{T_{hi}-T_{co}}\right)}\right] \quad \text{(A2)}$$

where U is the overall heat transfer coefficients based on the heat exchange surface area A. For the parallel-flow configuration, the total heat transfer rate can be expressed as

$$\dot{Q} = UA\left[\frac{(T_{ho}-T_{co})-(T_{hi}-T_{ci})}{\ln\left(\dfrac{T_{ho}-T_{co}}{T_{hi}-T_{ci}}\right)}\right]. \quad \text{(A3)}$$

Under the condition of $C_h \neq C_c$, the fluid temperatures at the flow outlets of a counterflow exchanger can be expressed as follows by manipulating equations (A1) and (A2):

$$T_{ho} = T_{hi}-(T_{hi}-T_{ci})\left\{\frac{1-\exp\left[UA\left(\dfrac{1}{C_c}-\dfrac{1}{C_h}\right)\right]}{1-\dfrac{C_h}{C_c}\exp\left[UA\left(\dfrac{1}{C_c}-\dfrac{1}{C_h}\right)\right]}\right\} \quad \text{(A4)}$$

$$T_{co} = T_{ci}+(T_{hi}-T_{ci})\left\{\frac{1-\exp\left[UA\left(\dfrac{1}{C_c}-\dfrac{1}{C_h}\right)\right]}{\dfrac{C_c}{C_h}-\exp\left[UA\left(\dfrac{1}{C_c}-\dfrac{1}{C_h}\right)\right]}\right\}. \quad \text{(A5)}$$

Similarly, outlet temperatures for parallel-flow heat exchangers can be expressed as

$$T_{ho} = T_{hi}-\frac{C_c}{(C_h+C_c)}(T_{hi}-T_{ci}) \times\left\{1-\exp\left[-UA\left(\frac{1}{C_c}+\frac{1}{C_h}\right)\right]\right\} \quad \text{(A6)}$$

$$T_{co} = T_{ci}+\frac{C_h}{(C_h+C_c)}(T_{hi}-T_{ci}) \times\left\{1-\exp\left[-UA\left(\frac{1}{C_c}+\frac{1}{C_h}\right)\right]\right\}. \quad \text{(A7)}$$

Although equations (A4)–(A7) have not appeared in the open literature, they can be easily verified by using the classical relationships between the heat exchanger effectivess and the number of transfer units.

Equations (A2), (A4) and (A5) were derived by using the assumption that C_h and C_c do not have the same value in the counterflow exchanger. For the special case of $C_h = C_c$ in a counterflow exchanger, these equations become singular. The governing equation for this case can be expressed as

$$\frac{dT_h}{dz} = \frac{dT_c}{dz} = -\frac{UA}{C_h}(T_h-T_c). \quad \text{(A8)}$$

From this expression, it can be shown that

$$T_h-T_c = D_1 = \text{constant}. \quad \text{(A9)}$$

By combining equation (A9) and (A8), one can determine the temperature distributions as

$$T_h = -\frac{UA}{C_h}D_1 z+D_2 \quad \text{(A10)}$$

$$T_c = -\frac{UA}{C_h}D_1 z-D_1+D_2. \quad \text{(A11)}$$

The integration constants, D_1 and D_2, can be determined as follows by letting $T_h = T_{hi}$ at $z = 0$ and $T_c = T_{ci}$ at $z = 1$:

$$D_1 = \frac{T_{hi}-T_{ci}}{\left(1+\dfrac{UA}{C_h}\right)} \quad \text{and} \quad D_2 = T_{hi}. \quad \text{(A12)}$$

Hence, the outlet temperatures can be determined by utilizing equations (A10), (A11) and (A12) to be

$$T_{ho} = T_{hi}-\frac{T_{hi}-T_{ci}}{\left(1+\dfrac{C_h}{UA}\right)} \quad \text{(A13)}$$

$$T_{co} = T_{hi}-\frac{T_{hi}-T_{ci}}{\left(1+\dfrac{UA}{C_h}\right)}. \quad \text{(A14)}$$

The total heat transfer rate therefore can be determined as

$$\dot{Q} = \frac{C_h(T_{hi}-T_{ci})}{\left(1+\frac{C_h}{UA}\right)}. \tag{A15}$$

The derivation of these closed-form relationships for this special case is to show its similarity to a three-channel counterflow heat exchanger having its split-flow heat capacity ratio being equal to unity.

APPENDIX B: HEAT TRANSFER ANALYSIS FOR A THREE-CHANNEL COUNTERFLOW EXCHANGER HAVING ITS SPLIT-FLOW HEAT CAPACITY RATIO BEING EQUAL TO UNITY

For a counterflow heat exchanger with its split-flow heat capacity rates, C_1 and C_s, being equal, equations (5) and (6) can be expressed as

$$\lambda \frac{dT_2}{dz} = -a_3(T_1-T_2) \tag{B1}$$

$$(1-\lambda)\frac{dT_3}{dz} = -a_2(T_1-T_3). \tag{B2}$$

Combining these two equations with equation (4), one obtains

$$\frac{d}{dz}[T_1-\lambda T_2-(1-\lambda)T_3] = 0. \tag{B3}$$

Hence, the temperature difference becomes

$$T_1-\lambda T_2-(1-\lambda)T_3 = D_1 \tag{B4}$$

where D_1 is an undetermined constant. Utilizing this equation to eliminate T_1 from equations (B1) and (B2), one can obtain

$$\lambda\frac{dT_2}{dz} = a_3[(1-\lambda)(T_2-T_3)-D_1] \tag{B5}$$

$$(1-\lambda)\frac{dT_3}{dz} = a_2[\lambda(T_3-T_2)-D_1]. \tag{B6}$$

Eliminating T_3 from equations (B5) and (B6), one obtains

$$\frac{d^2T_2}{dz^2} - v\frac{dT_2}{dz} = \frac{a_2a_3}{\lambda(1-\lambda)}D_1 \tag{B7}$$

where

$$v \equiv \frac{\lambda a_2}{(1-\lambda)} + \frac{(1-\lambda)a_3}{\lambda}. \tag{B8}$$

The general solution of this differential equation can be expressed as

$$T_2 = \frac{D_2}{v}e^{vz} - \frac{a_2a_3D_1}{v\lambda(1-\lambda)}z + D_3 \tag{B9}$$

where D_2 and D_3 are undetermined constants. Utilizing equations (B4) and (B5), one can express the temperature distributions in channels no. 1 and no. 3 as

$$T_1 = \left(\frac{1}{v}-\frac{\lambda}{a_3}\right)D_2e^{vz} - \frac{a_2a_3D_1}{v\lambda(1-\lambda)}z + D_3 + \frac{a_2D_1}{v(1-\lambda)} \tag{B10}$$

$$T_3 = \left[\frac{1}{v}-\frac{\lambda}{(1-\lambda)a_3}\right]D_2e^{vz} - \frac{a_2a_3D_1}{v\lambda(1-\lambda)}z + \left[\frac{a_2}{v(1-\lambda)^2}-\frac{1}{(1-\lambda)}\right]D_1 + D_3. \tag{B11}$$

The undetermined constants, D_1, D_2 and D_3, can be determined by specifying temperatures at inlets to be inlet temperatures. Hence, the three boundary conditions can be expressed as

$$\text{(i) At } z = 0: \quad T_1 = T_{1i} \tag{B12}$$

$$\text{(ii) At } z = 1: \quad T_2 = T_{2i} = T_{si} \tag{B13}$$

$$\text{(iii) At } z = 1: \quad T_3 = T_{3i} = T_{si}. \tag{B14}$$

Utilizing these three conditions, one can determine the coefficients in equations (B9), (B10) and (B11) to be

$$D_1 = \frac{T_{1i}-T_{2i}+(1-\lambda)(T_{2i}-T_{3i})\left[e^{-v}+\frac{a_3}{v\lambda}(1-e^{-v})\right]}{\left[1-\frac{1}{e^v}\right]\left[\frac{a_2}{v(1-\lambda)}+\frac{a_3}{v\lambda}-\frac{a_2a_3}{v^2\lambda(1-\lambda)}\right]+\frac{a_2a_3}{v\lambda(1-\lambda)}+\frac{1}{e^v}} \tag{B15}$$

$$D_2 = \frac{a_3}{\lambda e^v}\left[(1-\lambda)(T_{2i}-T_{3i})+\frac{a_2D_1}{v(1-\lambda)}-D_1\right] \tag{B16}$$

$$D_3 = T_{1i} - \frac{a_2D_1}{v(1-\lambda)} + D_2\left(\frac{\lambda}{a_3}-\frac{1}{v}\right). \tag{B17}$$

Each outlet temperatures of the three individual channels can then be evaluated by using equations (B9), (B10) and (B11) to be

$$T_{1o} = \frac{a_2D_1}{v(1-\lambda)}\left(1-\frac{a_3}{\lambda}\right) + D_2e^v\left(\frac{1}{v}-\frac{\lambda}{a_3}\right) + D_3 \tag{B18}$$

$$T_{2o} = \frac{D_2}{v} + D_3 \tag{B19}$$

$$T_{3o} = \left[\frac{a_2}{v(1-\lambda)}-1\right]\frac{D_1}{(1-\lambda)} + \left[\frac{1}{v}-\frac{\lambda}{(1-\lambda)a_3}\right]D_2 + D_3. \tag{B20}$$

The total heat transfer rate can then be evaluated as

$$\dot{Q} = C_1(T_{1i}-T_{1o}). \tag{B21}$$

The effectiveness of the heat exchanger can also be evaluated from equations (19) and (B21) to be

$$\varepsilon = \frac{T_{1i}-T_{1o}}{T_{1i}-T_{si}}. \tag{B22}$$

The number of transfer unit for this case can be determined by using equations (14) and (20) as

$$Ntu = \frac{T_{1i}-T_{1o}}{D_1}. \tag{B23}$$

Under the special-case condition as shown in equation (22), it can be shown that

$$v = \frac{a_3}{\lambda} = \frac{a_2}{(1-\lambda)}. \tag{B24}$$

The flow-distribution ratio under this condition can be determined as

$$\lambda = \frac{a_3}{a_2+a_3}. \tag{B25}$$

Hence, the temperature distributions in all three channels reduce to

$$T_1 = -vD_1z + D_1 + D_3 \tag{B26}$$

$$T_2 = \frac{D_2}{v}e^{vz} - vD_1z + D_3 \tag{B27}$$

$$T_3 = \left[1-\frac{1}{(1-\lambda)}\right]\frac{D_2}{v}e^{vz} - vD_1 z + D_3. \quad \text{(B28)}$$

For the split-flow condition of $T_{2i} = T_{3i} = T_{si}$, it can be shown by using equations (B13) and (B14) that $D_2 = 0$. Therefore, T_2 and T_3 have the same value everywhere in the heat exchanger. The non-zero coefficients in the above equations become

$$D_1 = \frac{T_{1i}-T_{si}}{1+v} \quad \text{(B29)}$$

$$D_3 = T_{2o} = T_{3o} = \frac{vT_{1i}+T_{si}}{1+v}. \quad \text{(B30)}$$

The outlet temperature of channel no.1 can then be determined as

$$T_{1o} = (1-v)D_1 + D_3 = \frac{vT_{si}+T_{1i}}{1+v}. \quad \text{(B31)}$$

APPENDIX C: DERIVATION OF THE ε-*Ntu* RELATIONSHIP FOR THREE-CHANNEL SPLIT-FLOW HEAT EXCHANGERS

The effectiveness of a three-channel split-flow heat exchanger and its equivalent number of transfer units are defined as those shown in equations (19) and (20), respectively. To derive their relationships for both the parallel-flow and the counterflow configurations, a differential length of the heat exchanger, dx, can be considered. The total rate of heat transfer through this differential area can be expressed as

$$\delta\dot{Q} = U_e P_e (T_1 - T_s)\,dx \quad \text{(C1)}$$
$$= U_{12}P_{12}(T_1-T_2)\,dx + U_{13}P_{13}(T_1-T_3)\,dx.$$

The heat is assumed to be transferred from the fluid in channel 1 to the split flows in channel 2 and channel 3 with an equivalent temperature, an equivalent heat transfer coefficient, and an equivalent perimeter being T_s, U_e, and P_e, respectively. However, the analysis can also be applied to the case with the direction of the heat transfer reversed, because simultaneous sign changes for temperature differences and heat transfer rates do not affect the final form of the governing equations. The equivalent split-flow temperature T_s can be related to temperatures of the fluid flows in the two side channels as

$$T_s = \frac{C_2 T_2 + C_3 T_3}{C_2 + C_3}. \quad \text{(C2)}$$

Substituting equation (C2) into equation (C1) and integrating the resulting equation from $x = 0$ to $x = L$, one can obtain

$$U_e P_e = \frac{C_1(a_3\theta + a_2\phi)}{L[\lambda\theta + (1-\lambda)\phi]} \quad \text{(C3)}$$

where all parameters are identical to those defined previously. Equation (61) can then be obtained by substituting equation (C3) into equation (20).

Assuming that there is no energy loss from the heat exchanger, one can express the overall heat-balance condition as

$$\dot{Q} = C_1(T_{1i} - T_{1o}) = C_s(T_{so} - T_{si}) \quad \text{(C4)}$$

where the inlet and the outlet temperatures of the central flow in channel no. 1 are denoted as T_{1i} and T_{1o}, respectively. Similarly, those of the equivalent split flow are also designated as T_{si} and T_{so}, respectively. The effectiveness of the heat exchanger can also be expressed as the following by substituting equation (C4) into equation (19):

$$\varepsilon \equiv \frac{\tau C_1}{\Delta C_{min}} \quad \text{(C5)}$$

where $\Delta \equiv T_{1i} - T_{si}$ and $\tau \equiv T_{1i} - T_{1o}$. Based on equations (C1), (C4), (C5) and (20), relationships between the effectiveness and the number of transfer units for both the parallel-flow and the counterflow configurations can be derived as shown below.

Parallel-flow exchangers

The heat-balance condition can be expressed as

$$\delta\dot{Q} = -C_1\,dT_1 = C_s\,dT_s. \quad \text{(C6)}$$

Combining equations (C1) and (C6), one can obtain

$$\frac{d(T_1 - T_s)}{(T_1 - T_s)} = -\left(\frac{1}{C_1} + \frac{1}{C_s}\right)U_e P_e\,dx. \quad \text{(C7)}$$

Integrating equation (C7) from $x = 0$ to $x = L$ and substituting equation (20) into the resulting expression, one obtains

$$\frac{\sigma}{\Delta} = \exp\left[-\left(\frac{1}{C_1} + \frac{1}{C_s}\right)Ntu\,C_{min}\right] \quad \text{(C8)}$$

where $\sigma \equiv T_{1o} - T_{so}$. Utilizing equation (C4), one can obtain

$$\sigma = \Delta - \left(1 + \frac{C_1}{C_s}\right)\tau. \quad \text{(C9)}$$

Substituting equation (C9) into equation (C5), one obtains

$$\varepsilon = \frac{C_1}{C_{min}}\left[\frac{1-\left(\frac{\sigma}{\Delta}\right)}{1+\left(\frac{C_1}{C_s}\right)}\right]. \quad \text{(C10)}$$

Consequently, the effectiveness can be expressed as follows by substituting equation (C8) into (C10):

$$\varepsilon = \frac{C_{max}}{(C_1 + C_s)}\left\{1 - \exp\left[-Ntu\frac{(C_1 + C_s)}{C_{max}}\right]\right\}. \quad \text{(C11)}$$

This expression is identical to the relationship shown in equation (31).

Counterflow exchangers

The heat-balance condition for the counterflow configuration can be expressed as

$$\delta\dot{Q} = -C_1\,dT_1 = -C_s\,dT_s. \quad \text{(C12)}$$

Combining equations (C1) and (C12), one obtains

$$\frac{d(T_1 - T_s)}{(T_1 - T_s)} = \left(\frac{1}{C_s} - \frac{1}{C_1}\right)U_e P_e\,dx. \quad \text{(C13)}$$

Integrating equation (C13) from $x = 0$ to $x = L$ and substituting equation (20) into the result, one obtains

$$\frac{\omega}{\eta} = \exp\left[\left(\frac{1}{C_s} - \frac{1}{C_1}\right)Ntu\,C_{min}\right] \quad \text{(C14)}$$

where $\omega \equiv T_{1o} - T_{si}$ and $\eta \equiv T_{1i} - T_{so}$. Based on the definition of temperature differences, one obtains

$$\omega = \Delta - \tau. \tag{C15}$$

In addition, the following relationship can also be obtained by using equation (C4):

$$\eta = \Delta - \tau\left(\frac{C_1}{C_s}\right). \tag{C16}$$

Solving Δ as well as τ from equations (C15) and (C16) and substituting them into equation (C5), one obtains

$$\varepsilon = \frac{C_1 C_s}{C_{min}}\left[\frac{1-\dfrac{\omega}{\eta}}{C_s - C_1\dfrac{\omega}{\eta}}\right]. \tag{C17}$$

Equation (C17) can be shown to be identical to equation (37a) by substituting equation (C14) into this equation. Equation (37b) can then be derived by applying L'Hospital's rule to equation (37a).